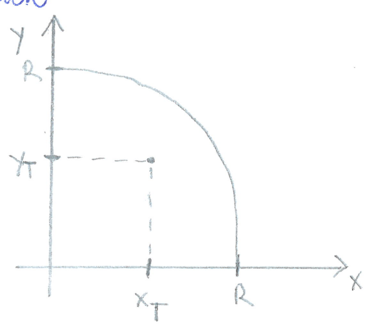
Příklad 8.1.

- čtvrtkruh



$$x_T = \frac{1}{M} \sum_i x_i m_i = \frac{1}{M} \int x \, dm$$

$$dm = \sigma \cdot dS = \frac{M}{S} dS$$

↑ plošná hustota

$$x_T = \frac{1}{S} \int x \, dS$$

$$y_T = \frac{1}{S} \int y \, dS$$

↓

polární souřadnice $\quad x = r \cos\varphi \qquad \varphi \in [0, \frac{\pi}{2}]$

$\quad y = r \sin\varphi \qquad r \in [0, R]$

$$dS = r \, dr \, d\varphi$$

$$\Rightarrow \quad x_T = \frac{1}{\frac{1}{4}\pi R^2} \int_0^{\pi/2} \int_0^R r \cos\varphi \; r \, dr \, d\varphi$$

$$x_T = \frac{4}{\pi R^2} \int_0^{\pi/2} \cos\varphi \, d\varphi \int_0^R r^2 \, dr$$

$$x_T = \frac{4}{\pi R^2} \left[\sin\varphi \right]_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^R = \frac{4}{\pi R^2} \cdot 1 \cdot \frac{R^3}{3}$$

$$x_T = \frac{4R}{3\pi}$$

podobně

$$y_T = \frac{1}{\frac{1}{4}\pi R^2} \int_0^{\pi/2} \int_0^R r \sin\varphi \; r \, dr \, d\varphi$$

$$y_T = \frac{4}{\pi R^2} \left[-\cos\varphi \right]_0^{\frac{\pi}{2}} \left[\frac{r^3}{3} \right]_0^R = \frac{4R}{3\pi}$$

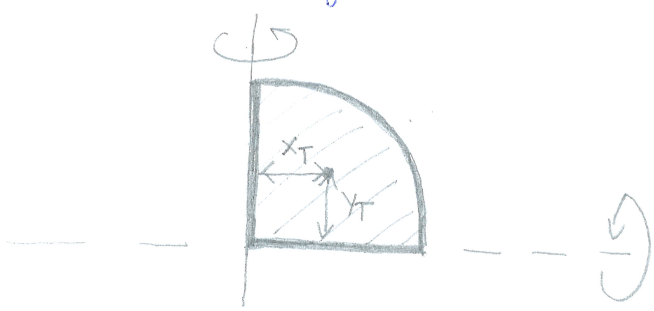
→ pomocí Pappovy věty

$$2\pi x_T S = V$$

plocha $\frac{1}{4}$ kruhu

rotační těleso vzniklé rotací $\frac{1}{4}$ kruhu

vzdálenost hmotného středu od osy otáčení



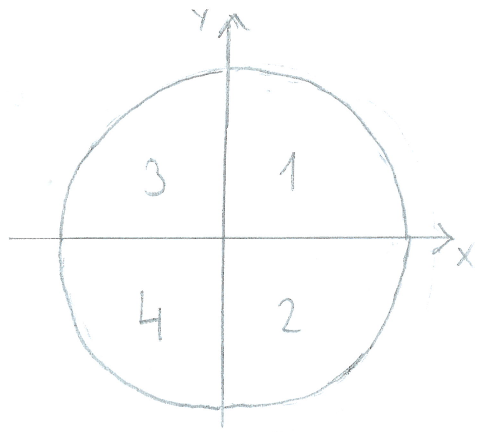
$$2\pi x_T \cdot \frac{1}{4}\pi R^2 = \frac{1}{2}\frac{4}{3}\pi R^3$$

$$x_T = \frac{4R}{3\pi}$$

$$y_T = \frac{4R}{3\pi}$$

stejné

• 4 čtvrtkruhy



hmotné středy čtvrtkruhů

1. $x_{T1} = \frac{4R}{3\pi} \cdot (+1)$
   $y_{T1} = \frac{4R}{3\pi} \cdot (+1)$
2. $x_{T2} = \frac{4R}{3\pi} \cdot (+1)$
   $y_{T2} = \frac{4R}{3\pi} \cdot (-1)$
3. $x_{T3} = \frac{4R}{3\pi} \cdot (-1)$
   $y_{T3} = \frac{4R}{3\pi} \cdot (+1)$
4. $x_{T4} = \frac{4R}{3\pi} \cdot (-1)$
   $y_{T4} = \frac{4R}{3\pi} \cdot (-1)$

$\downarrow$

hmotný střed kotouče

$$x_T = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{m_1 x_{T1} + m_2 x_{T2} + m_3 x_{T3} + m_4 x_{T4}}{m_1 + m_2 + m_3 + m_4}$$

$$x_T = \frac{1 \cdot \left(\frac{4R}{3\pi}\right) + 2 \cdot \left(\frac{4R}{3\pi}\right) + 3 \cdot \left(-\frac{4R}{3\pi}\right) + 4 \cdot \left(-\frac{4R}{3\pi}\right)}{1+2+3+4} = \frac{4R}{3\pi} \; \frac{1+2-3-4}{1+2+3+4} = \frac{4R}{3\pi} \; \frac{-4}{10}$$

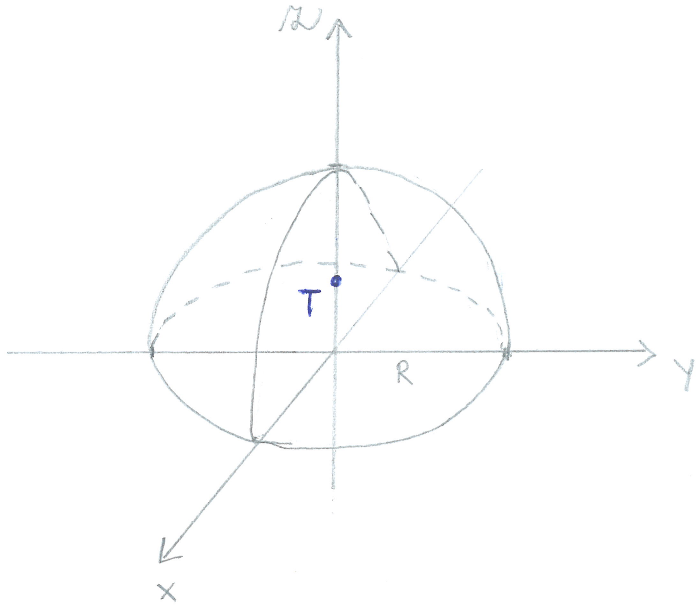
$$\underline{\underline{x_T = -\frac{8R}{15\pi}}}$$

$$y_T = \frac{\sum_i m_i y_i}{\sum_i m_i} = \frac{m_1 y_{T1} + m_2 y_{T2} + m_3 y_{T3} + m_4 y_{T4}}{m_1 + m_2 + m_3 + m_4}$$

$$y_T = \frac{1 \cdot \left(\frac{4R}{3\pi}\right) + 2\left(-\frac{4R}{3\pi}\right) + 3\left(\frac{4R}{3\pi}\right) + 4\left(-\frac{4R}{3\pi}\right)}{1+2+3+4} = \frac{4R}{3\pi} \cdot \frac{1-2+3-4}{1+2+3+4} = \frac{4R}{3\pi} \; \frac{-2}{10}$$

$$\underline{\underline{y_T = -\frac{4R}{15\pi}}}$$

Příklad 8.2.



- ze symetrie $x_T = 0$, $y_T = 0$
- počítáme pouze $z_T = \frac{1}{V} \int\limits_{\text{polokoule}} z \, dV$
- sférické souřadnice

$$x = r \sin\Theta \cos\varphi$$
$$y = r \sin\Theta \sin\varphi$$
$$z = r \cos\Theta$$

$$dV = r^2 \sin\Theta \, dr \, d\Theta \, d\varphi$$

- meze $r \in [0, R]$, $\Theta \in [0, \frac{\pi}{2}]$ ... polokoule, $\varphi \in [0, 2\pi]$

$$\Rightarrow z_T = \frac{1}{\frac{1}{2} \cdot \frac{4}{3}\pi R^3} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R \underbrace{r \cos\Theta}_{z} \underbrace{r^2 \sin\Theta \, dr \, d\Theta \, d\varphi}_{dV}$$

$$z_T = \frac{3}{2\pi R^3} \int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin\Theta \cos\Theta \, d\Theta \int_0^R r^3 \, dr$$

8.2.2.

$$r_T = \frac{3}{2\pi R^3} \left[\varphi \right]_0^{2\pi} \left[\frac{1}{2} \sin^2 \Theta \right]_0^{\pi/2} \left[\frac{r^4}{4} \right]_0^R$$

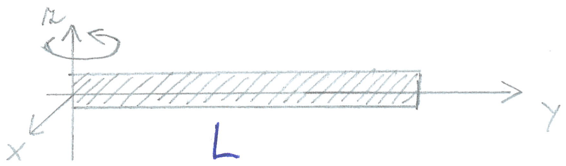
$$r_T = \frac{3}{\cancel{2\pi} \cancel{R^3}} \cancel{2\pi} \cdot \frac{1}{2} \cdot \frac{\cancel{R^4}}{4} = \underline{\underline{\frac{3}{8} R}}$$

Příklad 8.3.

moment setrvačnosti:
$$I = \sum_i m_i r_{i,\perp}^2 = \int r_\perp^2 \, dm$$

$$I = \frac{M}{V} \int_V r_\perp^2 \, dV$$

a) tyč



• kartézské souřadnice $\qquad dV = dx\,dy\,dz$

$\circledast$ rozměry v x-ovém (a) a z-ovém (b) směru zanedbatelné vzhledem k L

• meze
$$x \in \left[-\frac{a}{2}, \frac{a}{2}\right]$$
$$y \in [0, L]$$
$$z \in \left[-\frac{b}{2}, \frac{b}{2}\right]$$

• kolmá vzdálenost od osy otáčení $\qquad r_\perp = y$

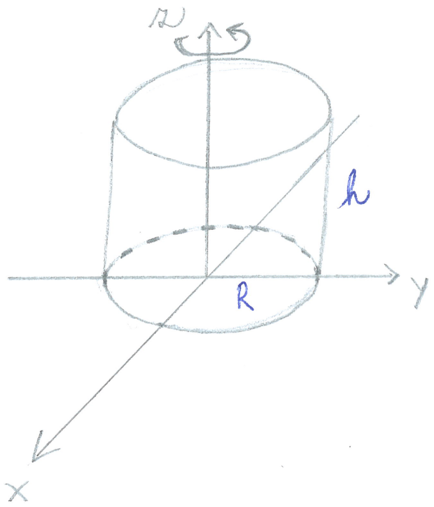
$$\Rightarrow I = \frac{M}{a \cdot b \cdot L} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_0^L \int_{-\frac{a}{2}}^{\frac{a}{2}} y^2 \, dx\,dy\,dz$$

$$I = \frac{M}{a \cdot b \cdot L} \left[z\right]_{-\frac{b}{2}}^{\frac{b}{2}} \left[\frac{y^3}{3}\right]_0^L \left[x\right]_{-\frac{a}{2}}^{\frac{a}{2}}$$

$$I = \frac{M}{a \cdot b \cdot L} \cdot b \cdot \frac{L^3}{3} \cdot a = \underline{\underline{\frac{1}{3} M L^2}}$$

$\circledast$ nebo rovnou: $\quad I = \dfrac{M}{L} \displaystyle\int_0^L y^2 \, dy = \nearrow$

b) válec



• cylindrické souřadnice

$$x = r\sin\varphi$$
$$y = r\cos\varphi$$
$$z = z$$

$$dV = r\,dr\,d\varphi\,dz$$

• meze

$$r \in [0, R]$$
$$\varphi \in [0, 2\pi]$$
$$z \in [0, h]$$

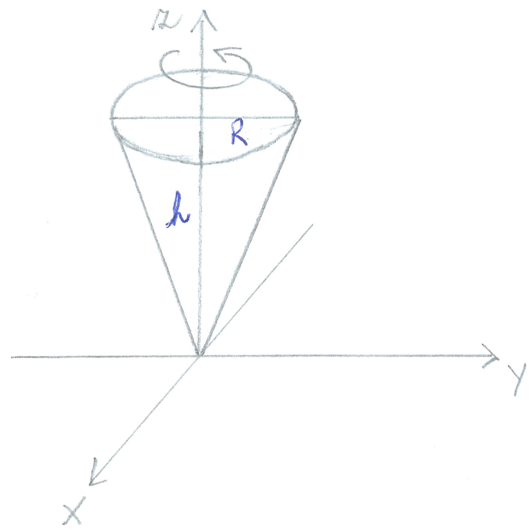
• kolmá vzdálenost od osy otáčení $r_\perp = r$

$$\Rightarrow I = \frac{M}{\pi R^2 h} \int_0^h \int_0^{2\pi} \int_0^R r^2\, r\,dr\,d\varphi\,dz$$

$$I = \frac{M}{\pi R^2 h} \left[z\right]_0^h \left[\varphi\right]_0^{2\pi} \left[\frac{r^4}{4}\right]_0^R$$

$$I = \frac{M}{\pi R^2 h}\, h \cdot 2\pi \cdot \frac{R^4}{4} = \underline{\underline{\frac{1}{2} M R^2}}$$

c) kužel



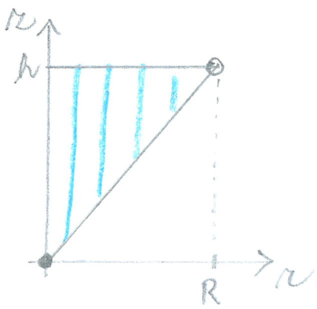
• cylindrické souřadnice

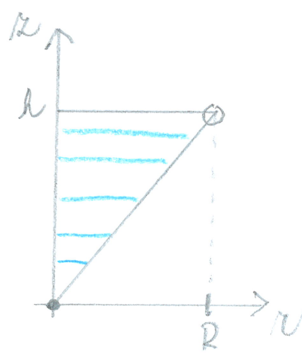
$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = z$$

$$dV = r\,dr\,d\varphi\,dz$$

• meze buď $r \in [0, R]$, $\varphi \in [0, 2\pi]$, $z \in [\frac{h}{R}r, R]$ nebo $r \in [0, \frac{R}{h}z]$, $\varphi \in [0, 2\pi]$, $z \in [0, h]$

→ proč? ... lineární závislost z a r





$$z = a \cdot r + b$$
$$0 = 0a + b \quad \Rightarrow b = 0$$
$$h = a \cdot R + b \quad \Rightarrow a = \frac{h}{R}$$

$$z = \frac{h}{R} r \quad \Longleftarrow$$
$$r = \frac{R}{h} z$$

• kolmá vzdálenost od osy otáčení $r_\perp = r$

první meze $\Rightarrow$

$$I = \frac{M}{\frac{1}{3}\pi R^2 h} \int_{\frac{h}{R}r}^{h} \int_0^{2\pi} \int_0^{R} r^2 \, r \, dr \, d\varphi \, dz$$

- nejprve integruji podle φ ... nezávislá mez

  poté integruji podle z ... mez závisí na r

  nakonec integruji podle r

$$I = \frac{3M}{\pi R^2 h} \left[\varphi\right]_0^{2\pi} \left[\int_0^R z \, r^3 \, dr \right]_{\frac{h}{R}r}^{h}$$

$$I = \frac{3M}{\pi R^2 h} 2\pi \int_0^R \left(h r^3 - \frac{h}{R} r^4\right) dr = \frac{6M}{R^2 h} \left[h \frac{r^4}{4} - \frac{h}{R} \frac{r^5}{5} \right]_0^R$$

$$I = \frac{6M}{R^2 h} \left(h \frac{R^4}{4} - h \frac{R^4}{5} \right) = \frac{6M}{R^2 h} h R^4 \frac{1}{20} = \underline{\underline{\frac{3}{10} M R^2}}$$

druhé meze $\Rightarrow$

$$I = \frac{M}{\frac{1}{3}\pi R^2 h} \int_0^{h} \int_0^{2\pi} \int_0^{\frac{R}{h}z} r^2 \, r \, dr \, d\varphi \, dz$$

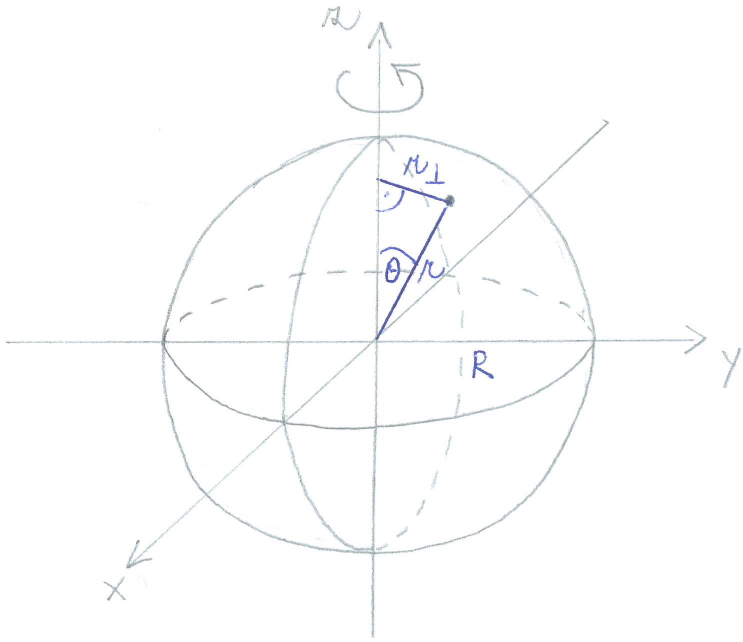
- nejprve integruji $\varphi \rightarrow$ potom $r \rightarrow$ potom z

$$I = \frac{3M}{\pi R^2 h} \left[\varphi\right]_0^{2\pi} \left[\int_0^h \frac{r^4}{4} \, dz \right]_0^{\frac{R}{h}z}$$

$$I = \frac{3M}{\pi R^2 h} 2\pi \int_0^h \left(\frac{R^4}{4h^4} z^4 \right) dz = \frac{6M}{R^2 h} \left[\frac{R^4}{4h^4} \frac{z^5}{5} \right]_0^h$$

$$I = \frac{6M}{R^2 h} \frac{R^4}{4h^4} \frac{h^5}{5} = \underline{\underline{\frac{3}{10} M R^2}}$$

d) koule (osa jdoucí těžištěm)



• sférické souřadnice

$$x = r \sin\theta \cos\varphi$$
$$y = r \sin\theta \sin\varphi$$
$$z = r \cos\theta$$

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

• meze
$$r \in [0, R]$$
$$\theta \in [0, \pi]$$
$$\varphi \in [0, 2\pi]$$

• kolmá vzdálenost od osy otáčení $\quad r_\perp = r \sin\theta$

$$\Rightarrow I = \frac{M}{\frac{4}{3}\pi R^3} \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin^2\theta \; r^2 \sin\theta \, dr \, d\theta \, d\varphi$$

$$I = \frac{3M}{4\pi R^3} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta \, d\theta \int_0^{R} r^4 \, dr$$

$$I = \frac{3M}{4\pi R^3} \left[\varphi\right]_0^{2\pi} \cdot \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} \cdot \left[\frac{r^5}{5}\right]_0^{R}$$

$$I = \frac{3M}{4\pi R^3} \cdot 2\pi \cdot \left(2 - \frac{2}{3}\right) \cdot \frac{R^5}{5} = \underline{\underline{\frac{2}{5} M R^2}}$$

8.3.6.

$$\int_0^{\pi} \sin^3\Theta \, d\Theta = \int_0^{\pi} \sin\Theta\,(1-\cos^2\Theta)\, d\Theta$$

$$= \int_0^{\pi} \left(\sin\Theta - \sin\Theta\cos^2\Theta\right) d\Theta = \left[-\cos\Theta + \frac{1}{3}\cos^3\Theta\right]_0^{\pi}$$

OK.

vnější funkce

derivace vnitřní funkce

$\Rightarrow$ ZKUS $(\cos^3\Theta)' = -3\cos^2\Theta\sin\Theta$

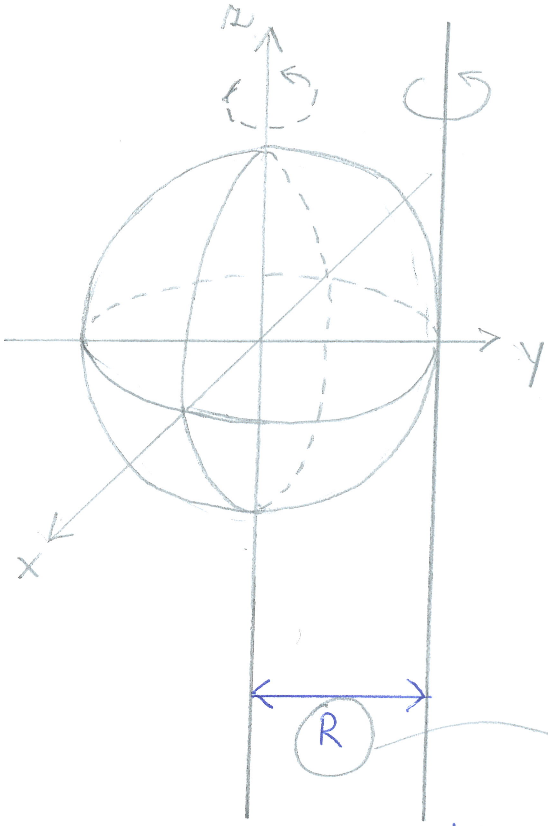
nebo

$$t = \cos\Theta$$
$$dt = -\sin\Theta\, d\Theta$$

$$\int_0^{\pi} (\sin\Theta - \sin\Theta\cos^2\Theta\, d\Theta) = \int_1^{-1} (-1+t^2)\, dt = \left[-t + \frac{1}{3}t^3\right]_1^{-1}$$

$$= \left[-\cos\Theta + \frac{1}{3}\cos^3\Theta\right]_0^{\pi}$$

e) koule (osa tečna koule)



o_T	o'
↓	↓
osa otáčení procházející těžištěm	osa otáčení vzdálená R
$J_T = \frac{2}{5} M R^2$	$J_{o'}$

→ Steinerova věta

$$J_{o'} = J_T + M R^2$$

$$J_{o'} = \frac{2}{5} M R^2 + M R^2 = \underline{\underline{\frac{7}{5} M R^2}}$$

Příklad 8.4.

Tenzor setrvačnosti :

$$I = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

$$I_{ij} = \frac{M}{V} \int \left(r^2 \delta_{ij} - x_i x_j \right) dV$$

$r^2 = x^2 + y^2 + z^2$

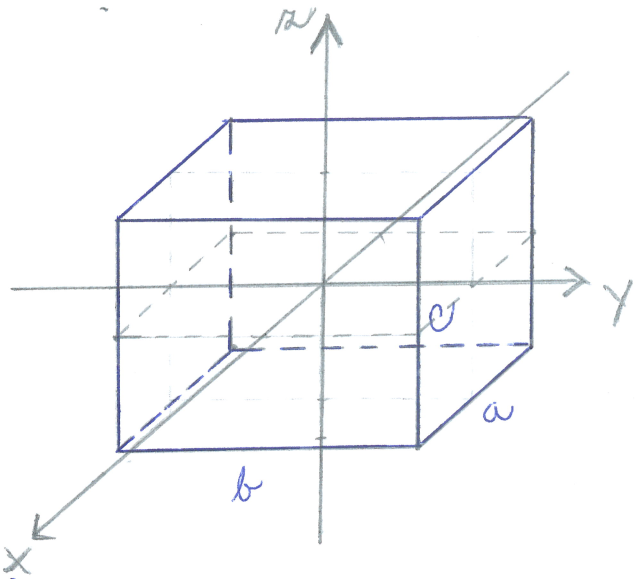
$x_1 = x \; ; \; x_2 = y \; ; \; x_3 = z$

KRONECKEROVO DELTA

$\delta_{ij} = 1 \iff i = j$

$\delta_{ij} = 0 \iff i \neq j$

kvádr (střed v počátku)



meze :

$x \in [-\frac{a}{2}, \frac{a}{2}]$
$y \in [-\frac{b}{2}, \frac{b}{2}]$
$z \in [-\frac{c}{2}, \frac{c}{2}]$

$$I_{11} = \frac{M}{abc} \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(\underbrace{x^2 + y^2 + z^2}_{r^2 \delta_{ij}} - \underbrace{x^2}_{x_i x_j} \right) dx\, dy\, dz$$

$$I_{11} = \frac{M}{abc} \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(y^2 + z^2 \right) dx\, dy\, dz$$

$$I_{11} = \frac{M}{abc} \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[x(y^2+z^2) \right]_{-\frac{a}{2}}^{\frac{a}{2}} dy\,dz$$

$$\uparrow x$$

$$I_{11} = \frac{M}{abc}\, a \int_{-\frac{c}{2}}^{\frac{c}{2}} \left[\frac{y^3}{3} + yz^2 \right]_{-\frac{b}{2}}^{\frac{b}{2}} dz$$

$$\uparrow y$$

$$I_{11} = \frac{M}{bc} \int_{-\frac{c}{2}}^{\frac{c}{2}} \left(\frac{b^3}{12} + bz^2 \right) dz = \frac{M}{bc} \left[\frac{b^3 z}{12} + b\frac{z^3}{3} \right]_{-\frac{c}{2}}^{\frac{c}{2}}$$

$$I_{11} = \frac{M}{bc} \left(\frac{b^3 c}{12} + \frac{bc^3}{12} \right) = \underline{\underline{\frac{1}{12} M (b^2 + c^2)}}$$

podobně

$$I_{22} = \frac{M}{V} \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+y^2+z^2-y^2) = \frac{M}{abc} \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+z^2)\, dx\,dy\,dz$$

$$= \underline{\underline{\frac{1}{12} M (a^2 + c^2)}}$$

$$I_{33} = \frac{M}{V} \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+y^2+z^2-z^2) = \frac{M}{abc} \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+y^2)\, dx\,dy\,dz$$

$$= \underline{\underline{\frac{1}{12} M (a^2 + b^2)}}$$

$$I_{12} = I_{21} = \frac{M}{abc} \int_{-\frac{c}{2}}^{\frac{c}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} -xy \; dx\,dy\,dz$$

$$= \frac{M}{abc} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left[-xyz\right]_{-\frac{c}{2}}^{\frac{c}{2}} dx\,dy$$

$\uparrow$ z

buď $\rightarrow$ $$= \frac{M}{abc}\, c \int_{-\frac{a}{2}}^{\frac{a}{2}} \left[-x\frac{y^2}{2}\right]_{-\frac{b}{2}}^{\frac{b}{2}} = 0$$

$\uparrow$ y

nebo

$$= \frac{M}{abc}\, c \int_{-\frac{b}{2}}^{\frac{b}{2}} \left[-\frac{x^2}{2}y\right]_{-\frac{a}{2}}^{\frac{a}{2}} = 0$$

podobně $I_{31} = I_{13} = 0$

$I_{23} = I_{32} = 0$

$$\Rightarrow \quad I = \frac{1}{12} M \begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

- diagonální tvar $\Rightarrow$ osy x, y, z jsou tzv. vlastní osy kvádru tělesová úhlopříčka

$\downarrow$

směr vektoru $\vec{\nu}^* = (a, b, c)$

$\downarrow$ jednotkový vektor

$$\vec{\nu} = \frac{\vec{\nu}^*}{|\vec{\nu}^*|} = \frac{(a, b, c)}{\sqrt{a^2+b^2+c^2}}$$

moment setrvačnosti vůči ose dané $\vec{\nu}$

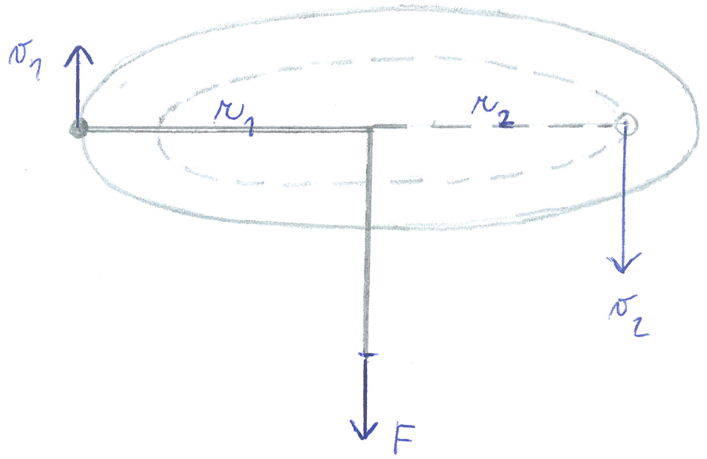
$$I_\nu = \sum_{i,j=1}^{3} \nu_i \nu_j I_{ij} = \frac{\frac{1}{12}M}{\left(\sqrt{a^2+b^2+c^2}\right)^2} \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$I_\nu = \frac{M}{12} \frac{1}{a^2+b^2+c^2} \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} ab^2+ac^2 \\ a^2b+bc^2 \\ a^2c+b^2c \end{pmatrix}$$

$$I_\nu = \frac{M}{12} \frac{1}{a^2+b^2+c^2} \left(a^2b^2 + a^2c^2 + a^2b^2 + b^2c^2 + a^2c^2 + b^2c^2 \right)$$

$$\underline{\underline{I_\nu = \frac{M}{6} \frac{a^2b^2+a^2c^2+b^2c^2}{a^2+b^2+c^2}}}$$

Příklad 8.5.



a) zákon zachování momentu hybnosti :

$$L_1 = L_2$$

$$I_1 \omega_1 = I_2 \omega_2$$

$$I = m r^2$$
$$\omega = v/r \quad \Longrightarrow \quad m r_1^2 \frac{v_1}{r_1} = m r_2^2 \frac{v_2}{r_2}$$

$$r_1 v_1 = r_2 v_2 \quad \Rightarrow \quad \underline{\underline{v_2 = v_1 \frac{r_1}{r_2}}}$$

b) $W = \Delta E_k = E_{k_2} - E_{k_1} = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = \underline{\underline{\frac{1}{2} m v_1^2 \left(\frac{r_1^2}{r_2^2} - 1 \right)}}$

$$= \frac{1}{2} I_2 \omega_2^2 - \frac{1}{2} I_1 \omega_1^2 = \frac{1}{2} m \cancel{r_2^2} \frac{v_2^2}{\cancel{r_2^2}} - \frac{1}{2} m \cancel{r_1^2} \frac{v_1^2}{\cancel{r_1^2}}$$

c) rovnováha tíhové a odstředivé síly :

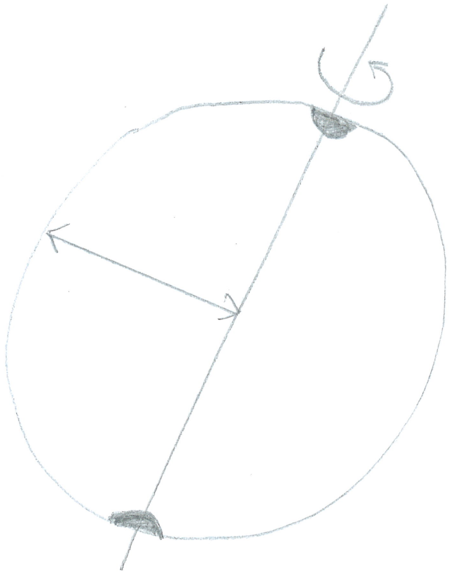
$$F_G = F_{od}$$

$$m_2 g = m \frac{v_2^2}{r_2}$$

$$m_2 = \frac{m}{g} \frac{v_2^2}{r_2}$$

$$\underline{\underline{m_2 = \frac{m}{g} \frac{v_1^2 r_1^2}{r_2^3}}}$$

Příklad 8.6



• led na pólech

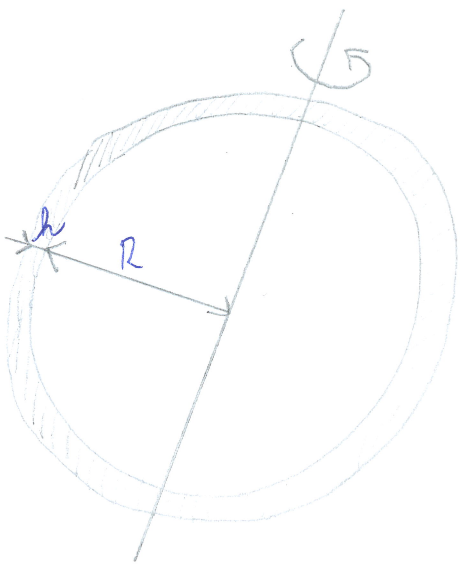
$\downarrow$

nulová vzdálenost od osy otáčení

$\Downarrow$

nepřispívá k momentu setrvačnosti

$$L_0 = I_0 \omega_0$$



• led roztaje

$\downarrow$

„původní Země" plus kulová slupka z vody o výšce h

$\Downarrow$

$$L_1 = I_1 \omega_1$$

$$I_1 = I_0 + \Delta I$$

$$\Delta I = \frac{M}{V}\int_{voda} r_\perp^2 \, dV = \rho_{H_2O} \int_0^{2\pi}\int_0^{\pi}\int_R^{R+h} \underbrace{r^2 \sin^2\Theta}_{r_\perp^2} \; \underbrace{r^2 \sin\Theta \, dr \, d\Theta \, d\varphi}_{dV}$$

$$\Delta I = \rho_{H_2O} \cdot 2\pi \cdot \frac{4}{3} \cdot \left[\frac{r^5}{5}\right]_R^{R+h} = \rho_{H_2O} \frac{8\pi}{3} \frac{1}{5}\left((R+h)^5 - R^5\right)$$

viz příklad 3d)

8.6.2.

$$\Delta I = \rho_{H_2O} \frac{8\pi}{3} \frac{1}{5} \left(\cancel{R^5} + 5R^4 h + 10 R^3 h^2 + 10 R^2 h^3 + 5 R h^4 + h^5 - \cancel{R^5} \right)$$

$$\Delta I = \rho_{H_2O} \frac{8\pi}{3} R^4 h \left(1 + \underbrace{2\frac{h}{R} + 2\frac{h^2}{R^2} + \frac{h^3}{R^3} + \frac{h^4}{5R^4}}_{\doteq 0} \right) \doteq \rho_{H_2O} \frac{8\pi}{3} R^4 h$$

$h = 61\,m$

$R = 6371000\,m$

$h << R$

• zákon zachování momentu hybnosti:

$$I_0 \omega_0 = I_1 \omega_1$$

$$I_0 \frac{2\pi}{T_0} = (I_0 + \Delta I) \frac{2\pi}{T_1}$$

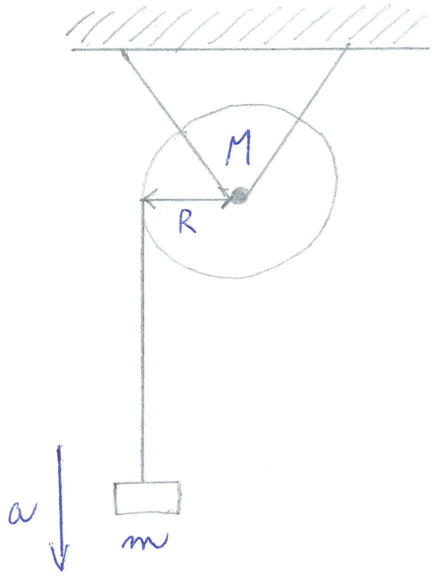
$$T_1 = T_0 \left(1 + \frac{\Delta I}{I_0}\right)$$

$$\Delta T = T_1 - T_0 = T_0 \frac{\Delta I}{I_0}$$

$$\Delta T = T_0 \frac{8\pi \rho_{H_2O} R^4 h}{3 I_0} \doteq 0{,}9\,s$$

1 den = 86400 s

Příklad 8.7.



i) přes energie (ZZE)

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

- $h = \frac{1}{2}at^2$
- $v = a\cdot t$
- $I = \frac{1}{2}MR^2$ (válec)
- $\omega = \frac{v}{R} = \frac{at}{R}$

$$\Rightarrow mg\cancel{\tfrac{1}{2}}\cancel{a}\cancel{t^2} = \cancel{\tfrac{1}{2}}ma^{\cancel{2}}\cancel{t^2} + \cancel{\tfrac{1}{2}}\frac{1}{2}M\cancel{R^2}\frac{a^{\cancel{2}}\cancel{t^2}}{\cancel{R^2}}$$

$$mg = ma + \frac{1}{2}Ma$$

$$a = \frac{m}{m+\frac{1}{2}M}g$$

obecně $$a = \frac{m}{m+I/R^2}g$$

ii) přes síly (2. N. Z / 2. věta impulzová)

$$m \cdot \vec{a} = \vec{F}_G + \vec{F}_v$$

$$m \cdot a = F_G - F_v$$

síla od válce

$$\vec{N}_v = I_v \vec{\varepsilon}$$

$$\vec{F}_v \times \vec{r} = I_v \vec{\varepsilon}$$

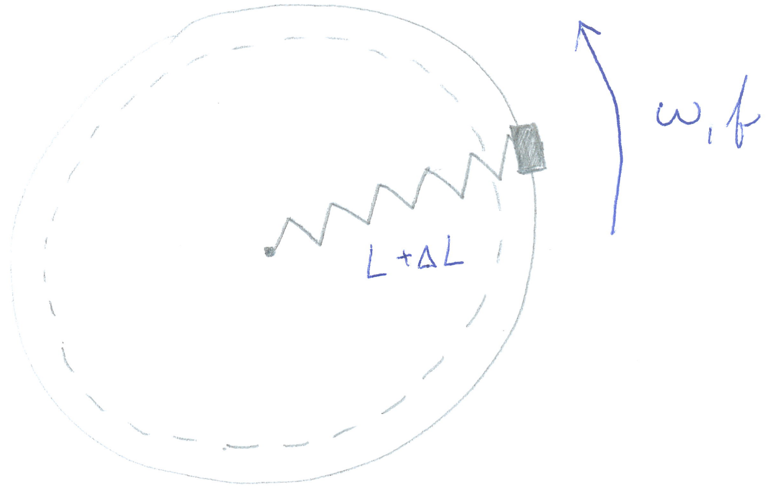
$$F_v \cdot R = I_v \frac{a}{R} \Rightarrow F_v = \frac{1}{2} M R^2 \frac{a}{R^2}$$

$$m \cdot a = m \cdot g - \frac{1}{2} M R^2 \frac{a}{R^2}$$

$$a = \frac{m}{m + \frac{1}{2} M} g$$

obecně $$a = \frac{m}{m + I/R^2} g$$

Příklad 8.8.



i) přes síly – rovnováha odstředivé síly a síly pružnosti

$$F_{od} = F_p$$

$$m\omega^2 (L + \Delta L) = k \cdot \Delta L$$

$$m\omega^2 L + m\omega^2 \Delta L = k \Delta L$$

$$\Delta L = L \frac{m\omega^2}{k - m\omega^2} = L \frac{1}{\frac{k}{m\omega^2} - 1}$$

$\omega = 2\pi f \Rightarrow$

$$\Delta L = \frac{1}{\frac{k}{4\pi^2 f^2 m} - 1}$$

- $k > m\omega^2 \Rightarrow \Delta L > 0$ prodloužení pružiny kvůli odstředivé síle
- $k = m\omega^2 \Rightarrow \Delta L \to \infty$ kritický stav, nekonečné prodloužení pružiny
- $k < m\omega^2 \Rightarrow$ těžko

ii) pomocí energií

- celková energie $E = \frac{1}{2} m v_r^2 + \frac{1}{2} I \omega^2 + \frac{1}{2} k (r - r_0)^2$

translace – pohyb pružiny (natahování a smršťování)

rotace

potenciální energie pružnosti

$$\left.\begin{aligned} * \; r - r_0 &= \Delta L \\ r_0 &= L \\ r &= L + \Delta L \end{aligned}\right\} \text{přeznačení}$$

$\Downarrow$

L značí (odteď) moment hybnosti

- moment hybnosti $L = I\omega$
(konstantní – zákon zachování momentu hybnosti)

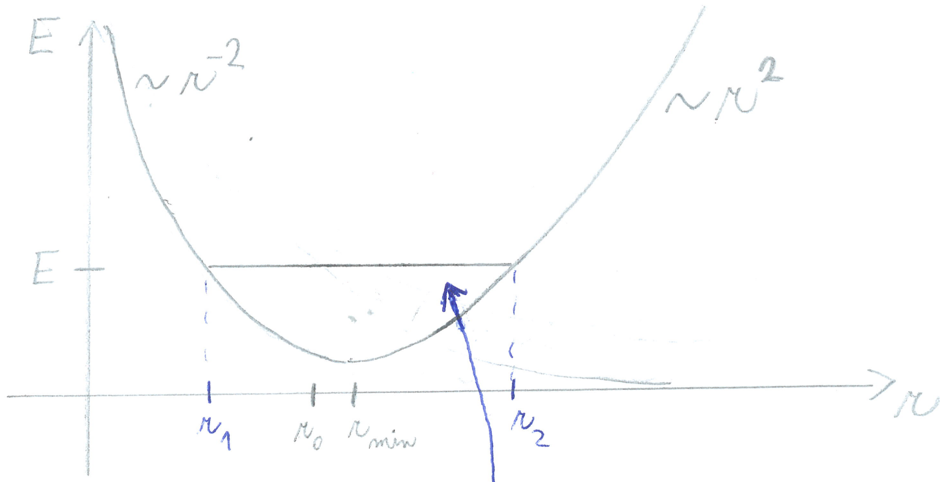
$$\Rightarrow \frac{1}{2} I \omega^2 = \frac{L^2}{2I}$$

- moment setrvačnosti $I = m r^2$

$$\Rightarrow E = \frac{1}{2} m v_r^2 + \underbrace{\frac{L^2}{2mr^2} + \frac{1}{2} k (r - r_0)^2}_{V_{eff}}$$

V_{eff} – efektivní potenciální energie

$$E - V_{eff} = \frac{1}{2} m v_r^2 \geq 0 \quad \Rightarrow \quad E \geq V_{eff}$$



pružina kmitá mezi délkami r_1 a r_2

↓

rovnovážný stav (pružina nekmitá)

$r_1 = r_2 = r_{min}$

↓

odpovídá minimu funkce $V_{eff}(r)$

• minimum V_{eff} : $\quad \frac{d}{dr}\left(\frac{L^2}{2mr^2} + \frac{1}{2}k(r-r_0)^2\right) = 0$

konstanta narozdíl od $\underline{\underline{I}}$ a $\underline{\underline{\omega}}$!

$$-\frac{L^2}{mr^3} + k(r-r_0) = 0$$

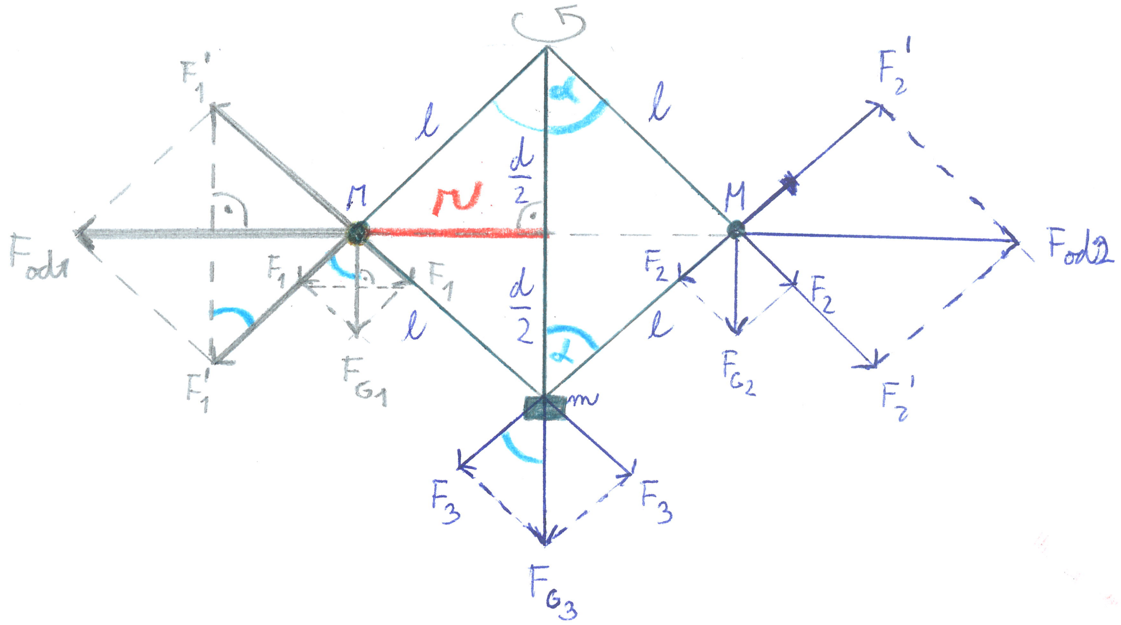
$$\frac{L^2}{mr^3} = k(r-r_0)$$

$$\frac{m^2 r^4 \omega^2}{mr^3} = k(r-r_0)$$

$$m r \omega^2 = k(r-r_0)$$

NEBOLI $\quad \underline{\underline{m\omega^2(L+\Delta L) = k\Delta L}}$ → stejně jako u sil

Příklad 8.9



rovnováha sil: $F_1' = F_1 + F_3$

$F_2' = F_2 + F_3$

symetrie
$F_{G1} = F_{G2} = F_G$
$F_1 = F_2 = F$
$F_{od1} = F_{od2} = F_{od}$
$F_1' = F_2' = F'$

síly

- $\dfrac{\frac{1}{2} F_{od}}{F'} = \sin\alpha$ $\qquad F_{od} = M r \omega^2 = M l \sin\alpha\, \omega^2$

$r = l \sin\alpha$

$\Rightarrow F' = \dfrac{1}{2} \dfrac{F_{od}}{\sin\alpha} = \dfrac{1}{2} M l \omega^2$

- $\dfrac{\frac{1}{2} F_G}{F} = \cos\alpha$ $\qquad F_G = Mg$

$\Rightarrow F = \dfrac{1}{2} \dfrac{F_G}{\cos\alpha} = \dfrac{1}{2} Mg \dfrac{1}{\cos\alpha}$

- $\dfrac{\frac{1}{2} F_{G3}}{F_3} = \cos\alpha$ $\qquad F_{G3} = mg$

$\Rightarrow F_3 = \dfrac{1}{2} \dfrac{F_{G3}}{\cos\alpha} = \dfrac{1}{2} mg \dfrac{1}{\cos\alpha}$

$$F' = F + F_3$$

$$\frac{1}{2} M l \omega^2 = \frac{1}{2} M g \frac{1}{\cos\alpha} + \frac{1}{2} m g \frac{1}{\cos\alpha}$$

$$M l \omega^2 \underbrace{\cos\alpha}_{} = (M+m) g$$

$$\cos\alpha = \frac{\frac{1}{2} d}{l} = \frac{d}{2l}$$

$$\frac{1}{2} M d \omega^2 = (M+m) g \quad \Rightarrow \quad M = \frac{m}{\frac{d\omega^2}{2g} - 1}$$

$$M = \frac{m}{\frac{2\pi^2 f^2 d}{g} - 1}$$

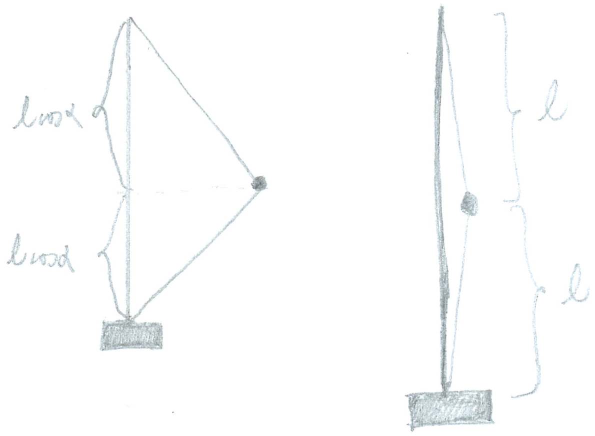
ii) pomocí energií

↳ podobně jako v příkladu 8.8.

$$V_{eff} = \frac{1}{2} I \omega^2 + mgh_3 + 2Mgh_{12}$$

- $\frac{1}{2} I \omega^2 = \frac{L^2}{2I} = \frac{L^2}{2 \cdot 2Mr^2} = \frac{L^2}{4Mr^2}$

- $h_3 = 2l - 2l\cos\alpha$



- $h_{12} = l - l\cos\alpha$

$$V_{eff} = \frac{L^2}{4Mr^2} + 2mgl(1-\cos\alpha) + 2Mgl(1-\cos\alpha)$$

$$V_{eff} = \frac{L^2}{4Ml^2\sin^2\alpha} + 2mgl(1-\cos\alpha) + 2Mgl(1-\cos\alpha)$$

$\uparrow$
$r = l\sin\alpha$

$$\frac{dV_{eff}}{d\alpha} = 0 = \frac{L^2}{4Ml^2\sin^3\alpha}(-2\cos\alpha) + 2mgl\sin\alpha + 2Mgl\sin\alpha$$

$$0 = -\frac{L^2 \cos\alpha}{2Ml^2 \sin^3\alpha} + 2gl\sin\alpha\,(m+M)$$

$$0 = -\frac{4Ml^4 \sin^4\alpha\,\omega^2 \cos\alpha}{2Ml^2 \sin^3\alpha} + 2gl\sin\alpha\,(m+M)$$

$\uparrow$

$$L = I\omega = 2Mr^2\omega = 2Ml^2\sin^2\alpha\,\omega$$

$$0 = 2l\sin\alpha\left(-Ml\omega^2\cos\alpha + mg + Mg\right)$$

$$\underline{Ml\omega^2\cos\alpha = mg + Mg}$$ $\rightarrow$ stejně jako u sil